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Quantum Affine Symmetry and Scattering Amplitudes of the Imaginary Coupled $d_4^{(3)}$ Affine Toda Field Theory

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Abstract

An exact S-matrix is conjectured for the imaginary coupled $d_4^{(3)}$ affine Toda field theory, using the $\mathcal{U}_q(g_2^{(1)})$ symmetry. It is shown that this S-matrix is consistent with the results for the case of real coupling using the breather-particle correspondence. For q a root of unity it is argued that the theory can be restricted to yield $\Phi(11|14)$ perturbations of WA_2 minimal models and the restriction is performed for the $(3, p')$ minimal models.

1 Introduction

Affine Toda field theories form a very important class of two-dimensional integrable field theories (for a review see [1]). In the imaginary coupling case, these theories provide natural generalizations of sine-Gordon theory and have solitonic excitations in their spectra. While in general these models are nonunitary as quantum field theories, their RSOS restrictions correspond to perturbations of W-symmetric rational conformal field theories (RCFTs), among them to unitary ones [2].

In the case of theories associated to simply-laced affine Lie algebras, the semi-classical mass ratios are stable under quantum corrections [3] and the S-matrices can be obtained using the fact that the theories are invariant under a quantum affine symmetry algebra of nonlocal charges. In the nonsimply-laced case, while the mass ratios are not stable under quantum corrections [4], it is again thought that the S-matrix can be obtained using the representation theory of the nonlocal symmetry algebra. We remark that the existing computations of the mass renormalization [4] in the nonsimply-laced case do not

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agree with each other and also that the instability of the classical solutions [5] casts a big question mark over the validity of the results in [3, 4]. However, it is still plausible that the mass ratios of the nonsimply-laced theories would be changed by quantum corrections similarly to the real coupling theories [6], but it is unclear how to make a consistent semiclassical quantisation in the imaginary coupling case.

A nice review of the concept of applying the quantum symmetry algebra to construct exact S-matrices can be found in [7]. In several cases the exact S-matrices have been computed: for $a_n^{(1)}$ affine Toda theory [8], for the $d_n^{(2)}$ [9] and the $b_n^{(1)}$ [10] case.

In this paper we will treat the imaginary coupled $d_4^{(3)}$ affine Toda field theory. The restrictions of this theory are argued to be connected with certain perturbations of WA_2 minimal models.

The layout of the paper is as follows: in Section 2 we briefly review the known facts about the quantum symmetry and the S-matrix. The derivation of the S-matrix is described in Section 3, together with an analysis of the bound state poles and the connection with the real coupling case via the breather-particle correspondence. Section 4 is devoted to the restriction to perturbed WA_2 minimal models, while in Section 5 we draw our conclusions. The paper ends with two Appendices (A and B), containing some formulas used in the main text.

2 Quantum symmetry and S-matrix

2.1 The quantum affine symmetry

Let us take an affine Lie algebra $\hat{g}$ and define the affine Toda field theory with the Lagrangian

$$ S = \int d^2 x \frac{1}{2} \partial_\mu \vec{\Phi} \partial_\mu \vec{\Phi} + \frac{\lambda}{2\pi} \int d^2 x \sum_{\vec{\alpha}_j} \exp\left(i\beta \frac{2}{(\vec{\alpha}_j, \vec{\alpha}_j)} \vec{\alpha}_j \cdot \vec{\Phi} \right) \ , \tag{2.1} $$

where the vectors $\vec{\alpha}_j$, $j = 0 \ldots r$ are the simple roots of $\hat{g}$ (meaning the simple roots of g plus the extending or affine root with label 0). The normalization of the roots is given by taking $(\vec{\alpha}_j, \vec{\alpha}_j) = 2$ for the long roots.

In the usual nomenclature, (2.1) is referred to as the $\hat{g}^\vee$ affine Toda action, where $\hat{g}^\vee$ denotes the affine Lie algebra dual to $\hat{g}$, whose roots $\vec{\gamma}_j$ are the coroots of $\hat{g}$

$$ \vec{\gamma}_j = \frac{2\vec{\alpha}_j}{(\vec{\alpha}_j, \vec{\alpha}_j)} \ . \tag{2.2} $$

It is immediately apparent that any simply-laced affine Lie algebra is self-dual, while for nonsimply-laced ones the dual is obtained by reversing the arrows in the Dynkin diagram. Our convention of the action follows [11] and is different from the one usually adopted in the literature. This was chosen for later convenience.

The theory (2.1) is known to be integrable. Besides the infinite number of commuting charges, however, there is a nonlocal non-abelian symmetry algebra,

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which is given by the quantum symmetry algebra $\mathcal{U}_q(\hat{g})$ [11, 12]. The parameter q is related to the coupling constant by

$$q = \exp\left(\frac{4\pi^2 i}{\beta^2}\right) \ . \tag{2.3}$$

For later convenience, we introduce another parametrization of the coupling constant:

$$\xi = \frac{\pi\beta^2}{8\pi - 3\beta^2} \ , \ q = \exp\left[i\pi\left(\frac{\pi}{2\xi} + \frac{3}{2}\right)\right] \ . \tag{2.4}$$

We suppose that $0 < \xi < \infty$, which means $0 < \beta^2 < 8\pi/3$. In analogy with sine-Gordon theory, the point $\beta^2 = 8\pi/3$ must be the coupling at which the interaction terms become irrelevant and the theory describes two free scalar fields at the infrared when $\beta^2 > 8\pi/3$. We will see later that this is consistent with the exact S-matrix and also with arguments using perturbed conformal field theory.

The restriction of the above theory to perturbed RCFT occurs when the parameter q is a root of unity and leads to the so-called RSOS scattering amplitudes.

To fix our conventions of the quantum affine Lie algebra, let me briefly summarize the defining relations. We define first the algebra $\mathcal{U}_q(\widetilde{\hat{g}})$, which is generated by elements $\{h_i, \ e_i, \ f_i, \ i = 0\ldots r\}$, satisfying the following commutation relations

$$[h_i, h_j] = 0, \ [h_i, e_j] = (\alpha_i, \alpha_j)e_j, \ [h_i, f_j] = -(\alpha_i, \alpha_j)f_j,$$
$$[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q_i - q_i^{-1}}, \ q_i = q^{(\alpha_i, \alpha_i)/2} \ , \tag{2.5}$$

together with the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1 - a_{ij}}{k}_{q_i} e_i^k e_j e_i^{1-a_{ij}-k} = 0 \ ,$$
$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1 - a_{ij}}{k}_{q_i} f_i^k f_j f_i^{1-a_{ij}-k} = 0 \ ,$$
$$i \neq j \ , \tag{2.6}$$

where

$$\binom{m}{k}_{q_i} = \frac{[m]_q!}{[k]_q![m - k]_q!}, \ [m]_q! = \prod_{1 \leq i \leq m}[i]_q, \ [i]_q = \frac{q^i - q^{-i}}{q - q^{-1}} \tag{2.7}$$

are the usual quantum binomial coefficients and

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{2.8}$$

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is the Cartan matrix of $g_2^{(1)}$. The coproduct is given by

$$\begin{aligned}
\Delta(e_i) &= q^{-h_i/2} \otimes e_i + e_i \otimes q^{h_i/2} \ , \\
\Delta(f_i) &= q^{-h_i/2} \otimes f_i + f_i \otimes q^{h_i/2} \ , \\
\Delta(q^{\pm h_i/2}) &= q^{\pm h_i/2} \otimes q^{\pm h_i/2} \ .
\end{aligned} \tag{2.9}$$

The conserved charges possess a definite Lorentz spin. Denoting the infinitesimal two-dimensional Lorentz generator by D we have

$$[D, e_i] = s_i e_i, \quad [D, f_i] = -s_i f_i, \quad [D, H_i] = 0, \quad i = 0, \ldots, r. \tag{2.10}$$

where s_i is the Lorentz spin of e_i. Adjoining the operator D to the algebra $\widetilde{\mathcal{U}_q(\hat{g})}$ results in the full algebra $\mathcal{U}_q(\hat{g})$.

Denoting the Lorentz spin of an operator A by $\mathrm{spin}(A)$, $\mathrm{spin} : \mathcal{U}_q(\hat{g}) \to \mathrm{R}$ is a gradation of $\mathcal{U}_q(\hat{g})$, which is uniquely fixed by giving $s_0, \ldots, s_r$. The change between the gradations can be performed with similarity transformations by exponentials of the Cartan elements h_i.

Denote the one-particle states by $|a, \alpha, \theta\rangle$, where a denotes the multiplet, α is the label within the multiplet and θ is the rapidity. The rapidity specifies the energy $E = m \cosh(\theta)$ and the momentum $p = m \sinh(\theta)$, where m denotes the mass of the particle. At fixed rapidity the particles in the multiplet a span the space V_a which carries a finite dimensional unitary representation π_a of $\widetilde{\mathcal{U}_q(\hat{g})}$, with zero central charge. Including the rapidity the one-particle space will be denoted as $V_a(\theta)$. Under a finite Lorentz transformation $L(\lambda) = \exp(\lambda D)$ the rapidity θ is shifted by λ

$$L(\lambda)|a, \theta\rangle = |a, \theta + \lambda\rangle. \tag{2.11}$$

From this we deduce that $V_a(\theta)$ carries the following infinite dimensional representation π of $\mathcal{U}_q(\hat{g})$:

$$\begin{aligned}
\pi_a^{(\theta)}(D) &= \frac{d}{d\theta}, \\
\pi_a^{(\theta)}(e_i) &= \pi_a(e_i) e^{s_i \theta}, \\
\pi_a^{(\theta)}(f_i) &= \pi_a(f_i) e^{-s_i \theta}, \\
\pi_a^{(\theta)}(h_i) &= \pi_a(h_i).
\end{aligned} \tag{2.12}$$

The action of the symmetry on asymptotic multi-particle states is given by the coproduct:

$$\pi_{a_1 \cdots a_n}^{(\theta_1 \cdots \theta_n)}(A) = (\pi_{a_1}^{(\theta_1)} \otimes \cdots \otimes \pi_{a_n}^{(\theta_n)}) \Delta^{n-1}(A), \tag{2.13}$$

where $\Delta^2 = (1 \otimes \Delta)\Delta, \quad \Delta^3 = (1 \otimes 1 \otimes \Delta)\Delta^2$, etc.

2.2 The two-particle S-matrices

The S-matrix gives the mapping of an incoming asymptotic two-particle state into an outgoing asymptotic two-particle state

$$S_{ab}(\theta - \theta') : \ V_a(\theta) \otimes V_b(\theta') \to V_b(\theta') \otimes V_a(\theta) \tag{2.14}$$

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$$|b, \beta', \theta'\rangle \otimes |a, \alpha', \theta\rangle = (S_{ab}(\theta - \theta'))^{\alpha'\beta'}_{\alpha\beta} \left(|a, \alpha, \theta\rangle \otimes |b, \beta, \theta'\rangle\right) \tag{2.15}$$

The quantum affine symmetry tells us that

$$S_{ab}(\theta - \theta')\pi^{(\theta\theta')}_{ab}(A) = \pi^{(\theta'\theta)}_{ba}(A)S_{ab}(\theta - \theta'), \qquad \forall A \in \widetilde{\mathcal{U}_q(\hat{g})}. \tag{2.16}$$

(2.16) means that $S_{ab}(\theta - \theta')$ is an intertwiner between the representation $\pi^{(\theta\theta')}_{ab}$ and the representation $\pi^{(\theta'\theta)}_{ba}$. Because these representations are irreducible for generic θ, θ', such an intertwiner is unique, up to an overall constant. This intertwiner is obtained by evaluating the universal R-matrix of $\widetilde{\mathcal{U}_q(\hat{g})}$ in the appropriate representation and gradation

$$\check{R}^{(s)}_{ab}(\theta - \theta') = P_{ab}\left((\pi^{(\theta)}_{s,a} \otimes \pi^{(\theta)}_{s,b})R\right) \tag{2.17}$$

and multiplying it by an overall scalar prefactor f_{ab},

$$S_{ab}(\theta - \theta') = f_{ab}(\theta - \theta')\check{R}^{(s)}_{ab}(\theta - \theta'). \tag{2.18}$$

Here $P_{ab}: V_a(\theta) \otimes V_b(\theta') \rightarrow V_b(\theta') \otimes V_a(\theta)$ is the permutation operator $P_{ab}: V_a \otimes V_b \mapsto V_b \otimes V_a$. The prefactor $f_{ab}(\theta)$ will be constrained by the requirements of unitarity, crossing symmetry and the bootstrap principle.

By definition, the universal R-matrix of $\widetilde{\mathcal{U}_q(\hat{g})}$ satisfies

$$R\Delta(A) = \Delta^{op}(A)R \qquad \forall A \in \widetilde{\mathcal{U}_q(\hat{g})}, \tag{2.19}$$

where Δ^{op} is the opposite coproduct obtained by interchanging the factors of the tensor product.

It can further be derived that multi-particle S-matrices are given by products of two-particle ones, the consistency of which is provided by the Yang-Baxter equation fulfilled by the universal R-matrix.

3 The S-matrix of the fundamental solitons in imaginary coupled $d_4^{(3)}$ Toda theory

The quantum symmetry of the $d_4^{(3)}$ theory is given by $\mathcal{U}_q(g_2^{(1)})$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \tag{3.1}$$

and the simple roots are given by

$$\alpha_0 = (-1/\sqrt{2}, -\sqrt{3/2}) \ ,$$
$$\alpha_1 = (\sqrt{2}, 0) \ , \ \alpha_2 = (-1/\sqrt{2}, 1/\sqrt{6}) \ . \tag{3.2}$$

There are two important subalgebras of $\mathcal{U}_q(g_2^{(1)})$: the generators $\{h_i, e_i, f_i, i = 1, 2\}$ form a subalgebra $\mathcal{A}_1$ isomorphic to $\mathcal{U}_q(g_2)$, while the algebra $\mathcal{A}_0$ generated by $\{h_i, e_i, f_i, i = 0, 1\}$ is isomorphic to $\mathcal{U}_q(a_2)$.

3.1 The R-matrix in the fundamental representation

We will assume that the fundamental solitons transform in the 7-dimensional representation of the algebra, which is also the fundamental representation of $\mathcal{A}_1$. In this space, the algebra $\mathcal{U}_q(\widehat{\hat{g}_2})$ is represented by the following matrices:

$$
h_0 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, h_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
h_2 = \frac{1}{3}\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},
$$

$$
e_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, e_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
e_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{q^2+1}{q}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{q^2+1}{q}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
f_i = e_i^{tr} \ , \ i = 0,1,2 \ . \tag{3.3}
$$

(^{tr} denotes usual matrix transposition.) In the following we will use the principal gradation, in which case all the rapidity dependence is shifted to the generators with index 0, i.e.

$$
\pi^{(\theta)}(h_i) = h_i \ , \ \pi^{(\theta)}(h_i) = x^{\delta_{i0}}e_i \ , \ \pi^{(\theta)}(h_i) = x^{-\delta_{i0}}f_i \ , \tag{3.4}
$$

where x is essentially the exponential of the rapidity (the precise correspondence will be given later).

We will solve the intertvining equation for the operator

$$
\widehat{\mathcal{R}}(x,q) = P_{12}\mathcal{R}(x,q) \ , \tag{3.5}
$$

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where $\mathcal{R}(x,q)$ denotes the universal R-matrix in the tensor product of two fundamental representations and x denotes the ratio x_1/x_2 of the spectral parameters in the first and second space, respectively.

The equations for the generators $X \in \mathcal{A}_1$ look like

$$[\widehat{\mathcal{R}}(x,q), \Delta(X)] = 0 . \tag{3.6}$$

This means that $\widehat{\mathcal{R}}(x,q)$ is a $\mathcal{U}_q(g_2)$ invariant operator in the space $\mathbf{7} \otimes \mathbf{7}$. This space decomposes into irreducible representations under $\mathcal{U}_q(g_2)$ in the following way

$$\mathbf{7} \otimes \mathbf{7} = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27} , \tag{3.7}$$

where we denoted irreducible representations of $\mathcal{U}_q(g_2)$ by their dimensions (this is unambigous for this case). So the general solution can be written as

$$\widehat{\mathcal{R}}(x,q) = \sum_{\mathbf{R}=\mathbf{1},\mathbf{7},\mathbf{14},\mathbf{27}} A_{\mathbf{R}}(x,q)\mathcal{P}_{\mathbf{R}} , \tag{3.8}$$

where $\mathcal{P}_{\mathbf{R}}$ denotes the projector onto the subspace corresponding to $\mathbf{R}$ and $A_{\mathbf{R}}(x,q)$ are scalar functions. The construction of the projectors is described briefly in Appendix A.

Note that since $h_0 = -2h_1 - 3h_2$ (corresponding to the fact that the level of the representation is 0), the intertwining equation for h_0 is satisfied automatically. Therefore we are left with the task of solving the intertwining equations corresponding to the step operators for the affine root α_0:

$$\widehat{\mathcal{R}}(x,q)(q^{-h_0/2} \otimes e_0 + xe_0 \otimes q^{h_0/2}) = (xq^{-h_0/2} \otimes e_0 + e_0 \otimes q^{h_0/2})\widehat{\mathcal{R}}(x,q) ,$$
$$\widehat{\mathcal{R}}(x,q)(q^{-h_0/2} \otimes f_0 + \frac{1}{x} f_0 \otimes q^{h_0/2}) = (\frac{1}{x}q^{-h_0/2} \otimes f_0 + f_0 \otimes q^{h_0/2})\widehat{\mathcal{R}}(x,q) .$$
$$\tag{3.9}$$

Once one of these equations is solved, the other one is satisfied automatically due to symmetry reasons. The solution of these equations (as well as the projectors) were computed using the computer algebra program MAPLE and is given by

$$A_{\mathbf{1}}(x,q) = \frac{1 - xq^{2/3}}{x - q^{2/3}} \frac{1 - xq^4}{x - q^4}$$

$$A_{\mathbf{7}}(x,q) = \frac{1 - xq^{8/3}}{x - q^{8/3}}$$

$$A_{\mathbf{14}}(x,q) = \frac{1 - xq^{2/3}}{x - q^{2/3}}$$

$$A_{\mathbf{27}}(x,q) = 1 \tag{3.10}$$

(where an overall normalization was chosen, since from the intertwining equation only the ratios of the amplitudes can be computed).

In (3.10) and also in several later expressions we encounter powers of $q^{1/3}$. One has to choose an appropriate branch of the third root function to ensure

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the analytic dependence of the amplitudes on the coupling constant. Let us make the choice

$$q^{1/3} = \exp\left(\frac{4\pi^2 i}{3\beta^2}\right) , \qquad (3.11)$$

and all powers of the form $q^{n/3}$ must be understood as the nth power of $q^{1/3}$, whenever n is an integer. This choice of the branch is justified later by the consistency of the results we obtain for the S-matrix. One can see that at the point $\beta^2 = 8\pi/3$ the scattering becomes trivial.

To verify the 'brute force' calculation used to obtain the projectors and the amplitudes, we can use the tensor product graph method [13]. In the above case, it says that the ratio of the amplitudes has to be

$$\frac{A_{\mathbf{R}_2}(x,q)}{A_{\mathbf{R}_1}(x,q)} = \left\langle \frac{1}{2}\left(C(\mathbf{R}_1) - C(\mathbf{R}_2)\right)\right\rangle ,$$

$$\langle a \rangle = \frac{x - q^a}{1 - xq^a} , \qquad (3.12)$$

whenever the representation $\mathbf{R}_2$ occurs in the tensor product of $\mathbf{R}_1$ with the representation corresponding to the affine root, which in our case is the adjoint representation **14**. $C(\mathbf{R})$ denotes the value of the quadratic Casimir of the classical (i.e. undeformed) group in the representation $\mathbf{R}$. Using the decomposition rules of $\mathcal{U}_q(g_2)$ and the values of the Casimir

$$C(\mathbf{1}) = 0 , \ C(\mathbf{7}) = 4 , \ C(\mathbf{14}) = 8 , \ C(\mathbf{27}) = 28/3 , \qquad (3.13)$$

it is easy to see that (3.10) and (3.12) are consistent. We remark that the quantum R-matrix of $\mathcal{U}_q(g_2^{(1)})$ has already appeared in connection with an exactly solvable 173-vertex model on the lattice [14].

In (3.10) the following pole singularities occur:

- $x = q^4$: the singular piece is proportional to the singlet projector. This corresponds to the occurence of the breathers as bound states of fundamental kinks.

- $x = q^{4/3}$: the pole is in the **7** channel, which gives higher kink multiplets and also the fundamental kink occuring as a bound state of fundamental kinks.

- $x = q^{2/3}$: this gives another bound state in the singlet+adjoint representation.

3.2 S-matrix for the fundamental solitons

The S-matrix can be built from $\widehat{\mathcal{R}}(x,q)$ using the principles of unitarity and crossing symmetry. $\widehat{\mathcal{R}}(x,q)$ satisfies the following relations:

$$\widehat{\mathcal{R}}(x,q)\widehat{\mathcal{R}}(1/x,q) = \mathrm{I} , \qquad (3.14)$$

$$\widehat{\mathcal{R}}(x,q)(q^4/x,q)\frac{(x-1)(x-q^{10/3})(x-q^{4/3})q^{4/3}}{(x-q^4)(x-q^{2/3})(x-q^{8/3})}$$

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$$= (C \otimes \mathrm{I})(P_{12} \widehat{\mathcal{R}}(x,q))^{t_1}(C \otimes \mathrm{I})P_{12} \ ,$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -q^{5/3} \\ 0 & 0 & 0 & 0 & 0 & q^{4/3} & 0 \\ 0 & 0 & 0 & 0 & -q^{1/3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1/3} & 0 & 0 & 0 & 0 \\ 0 & q^{-4/3} & 0 & 0 & 0 & 0 & 0 \\ -q^{-5/3} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ . \quad (3.15)$$

(3.14) is the unitarity property, while (3.15) shows that $\widehat{\mathcal{R}}(x,q)$ is almost crossing symmetric. The fact that the crossing transformation is given by $\theta \to i\pi - \theta$, is consistent with the following rapidity dependence of x

$$x = \exp\left[\left(\frac{4\pi}{\beta^2}h - h^\vee\right)\theta\right] = \exp\left(\frac{2\pi}{\xi}\theta\right) \ , \quad (3.16)$$

where $h = 4$ abd $h^\vee = 6$ are the Coxeter and dual Coxeter numbers of g_2. The matrix C given by (3.15) is the charge conjugation matrix in the homogeneous gradation. We remark that one has to be careful with the above notion of unitarity because it is not the usual notion of quantum field theory (QFT). Indeed, the $d_4^{(3)}$ affine Toda field theory is not expected to be a unitary field theory. This nonunitarity is usually reflected in the incorrect sign of the residues at the bound state poles, but there may be even more severe violations of QFT unitarity, connected with a notion of "pseudo-unitarity" in a theory with indefinite metric on the space of states [15]. In this paper, the term "unitarity" will mostly refer to relations of the type (3.14).

The universal R-matrix has the following additional properties:

$$(\mathcal{P}_{\mathbf{1}})_{12} \, \mathcal{R}_{13}(xq^2,q)\mathcal{R}_{23}(x/q^2,q) \, (\mathcal{P}_{\mathbf{1}})_{12} =$$
$$\frac{(xq^{2/3}-1)(x-q^{4/3})(xq^2-1)}{(x-q^{2/3})(xq^{4/3}-1)(x-q^2)} \, (\mathcal{P}_{\mathbf{1}})_{12} \otimes \mathbf{I}_3 \ , \quad (3.17)$$

$$(\mathcal{P}_{\mathbf{7}})_{12} \, \mathcal{R}_{13}(xq^{4/3},q)\mathcal{R}_{23}(x/q^{4/3},q) \, (\mathcal{P}_{\mathbf{7}})_{12} =$$
$$\frac{(xq^{4/3}-1)(x-q^{2/3})}{(x-q^{4/3})(xq^{2/3}-1)}\mathcal{R}_{(12)3}(x,q) \ , \quad (3.18)$$

where (12) denotes the seven-dimensional irreducible subspace in the tensor product of the first and second spaces. (3.17) gives the bootstrap relation for the kink–breather and (3.18) for the kink–higher kink scattering amplitudes. These formulae were computed by directly performing the matrix multiplications in MAPLE. They can be checked in the following easy way: from the left hand side of these equations, one can get the crossing transformation properties and the allowed poles of the right hand side expressions, which agree with the ones derived directly from the right hand side.

We can change the gradation by applying the following similarity transformation

$$A \to x^{\frac{3}{4}h_1+\frac{5}{4}h_2}Ax^{-\frac{3}{4}h_1-\frac{5}{4}h_2}, \ A = h_i, \ e_i, \ f_i, \quad (3.19)$$

and, correspondingly

$$P_{12}\widehat{\mathcal{R}}(x,q)_{spin} = x_1^{\frac{3}{4}h_1+\frac{5}{4}h_2} \otimes x_2^{\frac{3}{4}h_1+\frac{5}{4}h_2} P_{12}\widehat{\mathcal{R}}(x,q)x_1^{-\frac{3}{4}h_1-\frac{5}{4}h_2} \otimes x_2^{-\frac{3}{4}h_1-\frac{5}{4}h_2} \quad (3.20)$$

In this way we end up with the spin gradation, in which we have a charge conjugation matrix of the form

$$C_{spin} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad (3.21)$$

It can be checked readily that the evaluation representation in the homogenous gradation is given by

$$\pi_{spin}^{(\theta)}(e_0) = x^{1/4}e_0 , \quad \pi_{spin}^{(\theta)}(e_1) = x^{1/4}e_1 , \quad \pi_{spin}^{(\theta)}(e_2) = x^{1/12}e_2 ,$$
$$\pi_{spin}^{(\theta)}(f_0) = x^{-1/4}f_0 , \quad \pi_{spin}^{(\theta)}(f_1) = x^{-1/4}f_1 , \quad \pi_{spin}^{(\theta)}(f_2) = x^{-1/12}e_2 ,$$
$$(3.22)$$

and therefore the spins are

$$s_0 = s_1 = \frac{4\pi}{\beta^2} - \frac{3}{2} , \quad s_2 = \frac{4\pi}{3\beta^2} - \frac{1}{2} . \quad (3.23)$$

So, in this gradation, the spins of the currents are the 'physical' ones as derived from the Lagrangian (2.1) using only the canonical commutation relations and the canonical energy-momentum tensor. Therefore for the theory (2.1) the spin gradation can be considered to be the relevant one. Note that in this gradation the charge conjugation matrix is coupling constant independent and does not contain any phases (only some signs).

To get the real S-matrix, we must modify $\widehat{\mathcal{R}}(x,q)_{spin}$ to be really crossing symmetric, while preserving unitarity. Let us remember that the solution to the intertwining equations was determined only up to a scalar function multiple. We can see that

$$S(\theta) = \widehat{\mathcal{R}}(x,q)_{spin}S_0(x) \quad (3.24)$$

will be crossing symmetric and unitary if $f(x)$ satisfies

$$S_0(x)S_0(1/x) = 1 ,$$
$$S_0(q^4/x) = S_0(x)\frac{(x-1)(x-q^{10/3})(x-q^{4/3})q^{4/3}}{(x-q^4)(x-q^{2/3})(x-q^{8/3})} , \quad (3.25)$$

which can be rewritten as

$$S_0(\theta)S_0(-\theta) = 1 ,$$
$$S_0(i\pi - \theta) =$$
$$\frac{\sinh\frac{\pi}{\xi}(\theta - i\pi)\sinh\frac{\pi}{\xi}\left(\theta - \frac{2i\pi}{3}\right)\sinh\frac{\pi}{\xi}\left(\theta - \frac{i\pi}{6} - \frac{i\xi}{2}\right)}{\sinh\frac{\pi}{\xi}\theta \sinh\frac{\pi}{\xi}\left(\theta - \frac{i\pi}{3}\right)\sinh\frac{\pi}{\xi}\left(\theta - \frac{5i\pi}{6} - \frac{i\xi}{2}\right)}S_0(\theta) . \quad (3.26)$$

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The solution to the equations (3.26) is not unique. We will choose a 'minimal' solution which means that we allow only poles corresponding to some degeneration of the S-matrix to a projector and their crossing symmetric partners (which correspond to the complementary projector). The unique function with this property is

$$S_0(\theta) = \prod_{k=0}^{\infty} \frac{(1)\left(\frac{2\pi}{\xi}\right)\left(\frac{\pi}{3\xi}\right)\left(1+\frac{5\pi}{3\xi}\right)\left(\frac{1}{2}+\frac{7\pi}{6\xi}\right)\left(\frac{1}{2}+\frac{5\pi}{6\xi}\right)}{\left(1+\frac{\pi}{\xi}\right)\left(\frac{\pi}{\xi}\right)\left(\frac{4\pi}{3\xi}\right)\left(1+\frac{2\pi}{3\xi}\right)\left(\frac{1}{2}+\frac{\pi}{6\xi}\right)\left(\frac{1}{2}+\frac{11\pi}{6\xi}\right)} \ , \quad (3.27)$$

where

$$(x) = \frac{\Gamma\left(x+\frac{2k\pi}{\xi}+\frac{i\theta}{\xi}\right)}{\Gamma\left(x+\frac{2k\pi}{\xi}-\frac{i\theta}{\xi}\right)} \quad (3.28)$$

$S_0(\theta)$ can be given an integral representation as follows:

$$S_0(\theta) = \exp\left[-i\int_{-\infty}^{+\infty} \frac{\sin k\theta \, \sinh\frac{1}{3}\pi k \left(\cosh\left(\frac{\pi}{6}-\frac{\xi}{2}\right)k - \frac{1}{2}\right)}{k\, \sinh\frac{\xi k}{2} \cosh\frac{\pi k}{2}} dk\right] \quad (3.29)$$

The transformation between the infinite product (3.27) and the integral (3.29) can be performed using the formula

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz}-e^{-t}}{1-e^{-t}}\right]\frac{dt}{t} \ , \quad (3.30)$$

which is valid for $\Re e\, z > 0$. This completes the derivation of the kink-kink scattering matrix.

3.3 Breather-kink and breather-breather scattering amplitudes

From the infinite product expansion (3.27) one can identify the poles and zeros of the scalar factor $S_0(\theta)$ and one can compute the pole structure of the total S-matrix with the help of (3.10). The bound state poles of the kink-kink S-matrix turn out to be:

- $\theta = i(\pi - m\xi)$, $m > 0$ integer: singlet breathers. The $m = 1$ case will be denoted by B_1; the $m > 1$ cases will be considered to be excited states of this breather and denoted by $B_1^{(m-1)}$.

- $\theta = i\left(\frac{2\pi}{3} - m\xi\right)$, $m \geq 0$ integer: higher kink multiplets and the fundamental kink occuring as a bound state of fundamental kinks. We label them by K_{m+1}.

- $\theta = i\left(\frac{\pi}{6} - \left(m+\frac{1}{2}\right)\xi\right)$, $m \geq 0$ integer: bound states in the singlet+adjoint representation, denoted by A_{m+1}.

Of course, there are also the images of the poles required by crossing symmetry.

Using the formulae (3.17), (3.18) and the integral formula for the scalar factor S_0 (3.29), the S-matrices for the first two types of bound states can be computed. (The computation is much more difficult for the third type, although in principle straightforward). Here we list only the amplitudes for the lowest lying breather with the fundamental kink and with itself:

$$S_{K_1 B_1} = \left\langle \frac{\pi}{2} + \frac{\xi}{2} \right\rangle_{K_1} \left\langle \frac{5\pi}{6} - \frac{\xi}{2} \right\rangle_{K_2} \left\langle -\frac{2\pi}{3} \right\rangle_{CDD_1}$$

$$S_{B_1 B_1} = \left\langle \xi \right\rangle_{B_1^{(1)}} \left\langle \frac{2\pi}{3} \right\rangle_{B_1} \left\langle -\frac{\pi}{3} + \xi \right\rangle_{B_2} \times$$

$$\left\langle \frac{\pi}{6} - \frac{\xi}{2} \right\rangle_{AB_1} \left\langle -\frac{\xi}{2} - \frac{\pi}{6} \right\rangle_{CDD_2} , \tag{3.31}$$

where we used the notation

$$\langle x \rangle = \frac{\sinh\theta + i\sin x}{\sinh\theta - i\sin x} . \tag{3.32}$$

The labels at the bottom of the blocks denote the bound states to which the corresponding poles belong. K_1 denotes the fundamental kink, and B_1 is the first breather originating from the pole in $S_{K_1 K_1}$ at $\theta = i\pi - i\xi$. $B_1^{(1)}$ is the excited state of this first breather, corresponding to the pole in $S_{K_1 K_1}$ at $\theta = i\pi - 2i\xi$. B_2 is a singlet bound state of two B_1's (a higher breather). The pole AB_1 is presumably a singlet bound state pole (a breather) originating from the first adjoint+singlet soliton multiplet, which we denoted by A_1. This is supported by the observation that for $\xi > \frac{\pi}{3}$ when A_1 is not in the physical region, the pole AB_1 goes out of the physical strip as well.

What about the block labelled with CDD_1? To understand the notation, suppose that $\xi > \frac{\pi}{3}$. In this case the A_1 pole is missing in the $S_{K_1 K_1}$ amplitude. Since all higher kink scattering matrices are proportional to the fundamental one, we get that there are only bound states in the singlet and fundamental representation. The spectrum very closely resembles that of the Zhiber-Mikhailov-Shabat (ZMS) model [16], with the difference that $d_4^{(3)}$ Toda theory has a seven-component kink instead of a three-component one.

The block CDD_1 is just the factor which makes all the difference between the kink-breather scattering amplitude of the ZMS model and the kink-breather scattering amplitude of the $d_4^{(3)}$ Toda theory. This block only has poles outside the physical strip and so it is just an 'innocent' CDD factor. The block indexed with CDD_2 in $S_{B_1 B_1}$ is similar in that the poles originating from this factor are outside the physical strip for all values of ξ for which the first breather pole exists, i.e. for $\xi < \pi$. Note that if $\xi > \pi$, the spectrum of the theory simplifies to the fundamental kink K_1.

The masses of the above particles are the following:

$$m_{K_1} = m , \ m_{K_{n+1}} = 2m\sin\left(\frac{\pi}{6} + n\frac{\xi}{2}\right) , \ n \geq 1 ,$$

$$m_{B_1} = 2m\sin\frac{\xi}{2} , \ m_{B_1^{(n)}} = 2m\sin\frac{(n+1)\xi}{2} , \ n \geq 1 ,$$

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$$m_{B_2} = 4m \sin \frac{\xi}{2} \sin \left(\frac{\pi}{3} + \frac{\xi}{2} \right) \,,$$

$$m_{A_n} = 2m \cos \left(\frac{\pi}{12} - (2n - 1)\frac{\xi}{4} \right) \,,$$

$$m_{AB_1} = 4m \cos \left(\frac{\pi}{12} - \frac{\xi}{4} \right) \sin \frac{\xi}{2} \,. \tag{3.33}$$

The mass formula for m_{AB_1} is analogous to those of the breathers orginating from the kinks K_1 and K_2.

To close the bootstrap for the generic case would require the amplitude for A_1. However, this multiplet is reducible, so there may be various mixings between the singlet and adjoint component, making it impossible to apply the simple-minded tensor product graph approach which is used to compute R-matrices in multiplicity-free representations. Therefore we do not investigate the bootstrap any further. We remark, however, that the spectrum is in agreement with the counting of the classical soliton solutions [17], which shows that there should be two different soliton species associated to the two fundamental representations of g_2, i.e. to the representation **7** and to the adjoint representation **14**. The above results imply that the adjoint soliton multiplet is extended by a singlet at the quantum level. Furthermore, the mass ratio of the solitons K_1 and A_1 agrees with the first-order corrections computed in [4]. The higher multiplets K_n and A_n (for $n > 1$) can be thought of as excited states of the fundamental solitons in analogy with the ideas presented in [18].

3.4   Connection to real coupling affine Toda field theory

Continuing the coupling β to imaginary values we get an affine Toda field theory with real coupling. By the particle – first breather correspondence principle, the S-matrix of the first breather must then become the amplitude for the fundamental particle of the real coupling case.

The real coupling $d_4^{(3)}$ and $g_2^{(1)}$ affine Toda field theories form a dual pair. The S-matrix was computed on the basis of the hypothesis of 'floating masses' and strong–weak coupling duality [19, 20, 21]. The S-matrix of the fundamental particle in $d_4^{(3)}$ can be written as

$$S_{11} = \left\langle \frac{2\pi}{H} \right\rangle \left\langle \frac{2\pi}{3} \right\rangle \left\langle \frac{2\pi}{H} - \frac{\pi}{3} \right\rangle \left\langle \frac{\pi}{3} - \frac{4\pi}{H} \right\rangle \left\langle -\frac{4\pi}{H} \right\rangle, \tag{3.34}$$

where

$$H = 6\frac{8\pi + 3\beta'^2}{4\pi + 3\beta'^2} \,, \tag{3.35}$$

and β' is the (real) coupling constant.

Putting $\beta' = i\beta$ we find the relation

$$\xi = \frac{\pi}{3} - \frac{4\pi}{H}, \tag{3.36}$$

and one can easily check that S_{11} in (3.34) and $S_{B_1 B_1}$ in (3.31) become identical. This is a strong evidence that the S-matrix computed above really corresponds

to the imaginary coupled $d_4^{(3)}$ affine Toda field theory. The second particle of the real coupling theory can be identified with AB_1, which was conjectured to be the lowest breather of the second soliton species. This is in accord with the assignment of the classical particles to the nodes of the Dynkin diagram of g_2.

4   Perturbed WA_2 minimal models

The action (2.1) can be rewritten as the action of a conformal A_2 Toda theory with a perturbation term:

$$S = S_{A_2} + S_{pert} \ ,$$

$$S_{A_2} = \int d^2x \frac{1}{2} \partial_\mu \vec{\Phi} \partial_\mu \vec{\Phi} + \frac{\lambda}{2\pi} \int d^2x \sum_{j=0}^{1} \exp\left(i\beta \frac{2}{(\vec{\alpha}_j, \vec{\alpha}_j)} \vec{\alpha}_j \cdot \vec{\Phi} \right) \ ,$$

$$S_{pert} = \frac{\lambda}{2\pi} \int d^2x \exp\left(i\beta \frac{2}{(\vec{\alpha}_2, \vec{\alpha}_2)} \vec{\alpha}_2 \cdot \vec{\Phi} \right) \ . \tag{4.1}$$

S_{A_2} corresponds to a WA_2-invariant conformal field theory with the central charge [22]

$$c = 2 \left(1 - 12 \left(\frac{\beta}{\sqrt{4\pi}} - \frac{\sqrt{4\pi}}{\beta} \right)^2 \right) \ . \tag{4.2}$$

When

$$\frac{\beta}{\sqrt{4\pi}} = \sqrt{\frac{p}{p'}} \ , \ p, \ p' \text{ coprime integers} \tag{4.3}$$

this is just the central charge corresponding to the (p, p') minimal model of the WA_2 algebra, which we denote with $WA_2(p, p')$. The field content of the minimal model can be described by giving the spectrum of the primary fields. These are labelled by four integers n_1, n_2, m_1, m_2 and denoted $\Phi(n_1 n_2 | m_1 m_2)$. To each of these fields one can associate a vector

$$\vec{\beta}(n_1 n_2 | m_1 m_2) = \sum_{i=1}^{2} (\alpha_-(1 - n_i) + \alpha_+(1 - m_i) \omega_i \ , \tag{4.4}$$

where ω_i are the fundamental weights of A_2 and we define

$$\alpha_+^2 = \sqrt{\frac{p}{p'}} \ , \ \alpha_- = -\frac{1}{\alpha_+} \ , \ \alpha = \alpha_+ + \alpha_- \ . \tag{4.5}$$

The conformal weight of the field $\Phi(n_1 n_2 | m_1 m_2)$ is [23]

$$h(n_1 n_2 | m_1 m_2) = \frac{1}{2} \vec{\beta}^2 - \alpha \vec{\rho} \vec{\beta} \ , \tag{4.6}$$

where

$$\vec{\rho} = \sum_{i=1}^{2} \omega_i \ . \tag{4.7}$$

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In our case the perturbing term turns out to be the field $\Phi(11|14)$. The weight of this field is

$$h(11|14) = 6\frac{p}{p'} - 3 \ . \tag{4.8}$$

To get a massive field theory, we require the perturbing field to be a relevant one, which means that its weight is less than one. Then we obtain the following condition:

$$\frac{p}{p'} < \frac{2}{3} \ \text{or} \ \beta^2 < 8\pi/3 \tag{4.9}$$

This is only statisfied for nonunitary minimal models of WA_2 ($|p - p'| > 1$).

For the above choice of β the parameter q becomes a root of unity. In the sine-Gordon and the ZMS model for these values of the couplings the state of space can be RSOS restricted to obtain integrable perturbations of the corresponding Virasoro minimal model ($\Phi_{(1,3)}$ perturbations from sine-Gordon [24], and $\Phi_{(1,2)}$ [16] or $\Phi_{(1,5)}$ [15] perturbations from the ZMS case). Following this analogy we assume that there exists a consistent restriction of the affine Toda field theory to the perturbed minimal model, which is described by the corresponding restriction of the representation theory of $\mathcal{U}_q(A_2)$ at this value of q. In view of the above discussion, we cannot expect to get a unitary quantum field theory after the restriction.

The condition (4.9) is in accord with what was said in connection with the special point $\beta^2 = 8\pi/3$ in subsection 2.1. If $\beta^2 > 8\pi/3$, the perturbation is irrelevant and flows back to the WA_2 minimal model in the infrared. In this regime the unrestricted theory is expected to be equivalent to two free bosons. Using the analogy with the Virasoro case we think that the restriction to the perturbed minimal model is equivalent to the Feigin-Fuchs free field construction of $WA_2(p, p')$ if the coupling of the perturbing term is set to zero.

To have this restriction we need to go over to a gradation in which all the rapidity dependence is carried by the generators corresponding to the root α_2. In this gradation there is a $\mathcal{U}_q(A_2)$ algebra acting on the space of states. The fundamental kink-kink scattering matrix can then be decomposed into irreducible parts under $\mathcal{U}_q(A_2)$. This is described in details in Appendix B.

One can see that for $q^6 = 1$ the singlet-singlet amplitude S_{11}^{11} is unitary in itself. For generic q, the unitarity equations demands the presence of the particles in $\mathbf{3}$ and $\bar{\mathbf{3}}$ as intermediate states. Therefore we can interpret this fact as the existence of a consistent restriction to the singlet sector. The corresponding perturbed minimal models are $WA_2(3, p') + \Phi(11|14)$. Using the expression for $\widehat{\mathcal{R}}(x, q)_{11}^{11}$ (B.3), we conjecture that the scattering in these models is obtained from the following fundamental amplitude:

$$\frac{\sinh \frac{\pi}{\xi}(\theta + i\pi)}{\sinh \frac{\pi}{\xi}(\theta - i\pi)} \frac{\sinh \frac{\pi}{\xi}\left(\theta + i\frac{2\pi}{3}\right)}{\sinh \frac{\pi}{\xi}\left(\theta - i\frac{2\pi}{3}\right)} S_0(\theta) \ , \tag{4.10}$$

where

$$\xi = \frac{3\pi}{2p' - 9} \ . \tag{4.11}$$

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The models $WA_2(3, p') + \Phi(11|14)$ are the analogs of the perturbed Virasoro minimal models $Vir(2, p') + \Phi(1, 5)$. In the latter case, the state space is again restricted to allow only singlet representation and from the kink transforming in the singlet+doublet representation only the singlet piece survives [15].

5   Conclusions

We have obtained the exact S-matrix of the imaginary coupled $d_4^{(3)}$ affine Toda field theory using the quantum affine symmetry of the model. This S-matrix proved to be consistent with the real coupling case using the breather-particle correspondence principle. It was argued that restricting the S-matrix in the A_2 gradation one should obtain exact S-matrices of the perturbed conformal field theories $WA_2(p, p') + \Phi(11|14)$. In the $p = 3$ case the fundamental amplitude was computed explicitly. We saw that the relevance of the perturbation restricts p, p' in such a way that unitary minimal theories are excluded.

To go further and obtain the S-matrix for the generic $WA_2(p, p') + \Phi(11|14)$ case, one would need an explicit expression for the $6 - j$–symbols of the algebra $\mathcal{U}_q(A_2)$, which, to our knowledge, is not yet available in general.

On the other hand, one can get unitary perturbations of $WA_2(p, p')$ by considering the operator $\Phi(11|12)$. This operator has conformal weight

$$h(11|12) = \frac{4}{3}\frac{p}{p'} - 1 \ , \tag{5.1}$$

and so it is relevant for the unitary case $|p - p'| = 1$. The corresponding affine Toda field theory is $g_2^{(1)}$ which has a symmetry algebra $\mathcal{U}_q(d_4^{(3)})$. The kinks in this case are in the representation $\mathbf{8}$ of the $\mathcal{U}_q(a_2)$ subalgebra of $\mathcal{U}_q(d_4^{(3)})$. Unfortunately, while this representation is irreducible, taking the tensor product of $\mathbf{8}$ with itself $\mathbf{8}$ occurs twice, making the representation theoretic computation of the S-matrix extremely difficult.

A motivation for computing these S-matrices is the fact that the restrictions to perturbed WA_2 minimal models can be checked using the Thermodynamical Bethe Ansatz (TBA) and Truncated Conformal Space Approach (TCSA). Although there are still some obstacles in the way to a real check, this is an interesting line of research to pursue, since it could provide a verification of S-matrices for imaginary-coupled nonsimply-laced affine Toda field theories.

On the semiclassical level, in such theories there are not enough solitons to fill up the affine algebra multiplets completely [25]. It was proposed that the semiclassical spectrum must be extended in the quantum field theory to form complete multiplets [8, 12]. A combined TCSA/TBA verification of the S-matrix proposed in this paper could lend strong support to this idea.

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Appendix

A   The invariant projectors

The calculation of the quantum group invariant projectors $\mathcal{P}_{\mathbf{R}}$ proceeds as follows. First we take the 49-dimensional space of the tensor product $\mathbf{7} \otimes \mathbf{7}$ of two fundamental representations. We can define the action of $\mathcal{U}_q(G_2)$ on this space using the coproduct:

$$H_i = \Delta(h_i), \ E_i = \Delta(e_i), \ F_i = \Delta(f_i), \ i = 1, 2. \tag{A.1}$$

Then we can find the highest weight vectors by computing the intersection of the kernels of E_i, $i = 1, 2$. This gives a four-dimensional space, which can be spanned by the highest weight vectors of the representations $\mathbf{1}$, $\mathbf{7}, \mathbf{14}$ and $\mathbf{27}$. These will be labelled by $|\mathbf{1} ; 0, 0\rangle$, $|\mathbf{7} ; 0, 1\rangle$, $|\mathbf{14} ; 1, 0\rangle$ and $|\mathbf{27} ; 2, 0\rangle$, respectively. The first number is the dimension of the representation, while the second two numbers are the Dynkin indices of the corresponding weight vector of the Lie algebra G_2. With these notations, we obtain the following expressions for the basis vectors of the representations $\mathbf{1}$, $\mathbf{7}$ and $\mathbf{14}$:

$$|\mathbf{14} ; 1, 0\rangle \ ,$$
$$|\mathbf{14} ; -1, 3\rangle = F_1|\mathbf{14} ; 1, 0\rangle \ ,$$
$$|\mathbf{14} ; 0, 1\rangle = F_2|\mathbf{14} ; -1, 3\rangle \ ,$$
$$|\mathbf{14} ; 1, -1\rangle = F_2|\mathbf{14} ; 0, 1\rangle \ ,$$
$$|\mathbf{14} ; -1, 2\rangle = F_1|\mathbf{14} ; 1, -1\rangle \ ,$$
$$|\mathbf{14} ; 2, -3\rangle = F_2|\mathbf{14} ; 1, -1\rangle \ ,$$
$$|\mathbf{14} ; 0, 0; 1\rangle = F_2|\mathbf{14} ; -1, 2\rangle \ ,$$
$$|\mathbf{14} ; 0, 0; 2\rangle = F_1|\mathbf{14} ; 2, -3\rangle \ ,$$
$$|\mathbf{14} ; -2, 3\rangle = F_1|\mathbf{14} ; 0, 0; 1\rangle \ ,$$
$$|\mathbf{14} ; 1, -2\rangle = F_2|\mathbf{14} ; 0, 0; 1\rangle \ ,$$
$$|\mathbf{14} ; -1, 1\rangle = F_2|\mathbf{14} ; -2, 3\rangle \ ,$$
$$|\mathbf{14} ; 0, -1\rangle = F_2|\mathbf{14} ; -1, 1\rangle \ ,$$
$$|\mathbf{14} ; 1, -3\rangle = F_2|\mathbf{14} ; 0, -1\rangle \ ,$$
$$|\mathbf{14} ; -1, 0\rangle = F_1|\mathbf{14} ; 1, -3\rangle \ ,$$
$$|\mathbf{7} ; 0, 1\rangle \ ,$$
$$|\mathbf{7} ; 1, -1\rangle = F_2|\mathbf{7} ; 0, 1\rangle \ ,$$
$$|\mathbf{7} ; -1, 2\rangle = F_1|\mathbf{7} ; 1, -1\rangle \ ,$$
$$|\mathbf{7} ; 0, 0\rangle = F_2|\mathbf{7} ; -1, 2\rangle \ ,$$

$$|\mathbf{7} \; ; \; 1, -2\rangle = F_2 |\mathbf{7} \; ; \; 0, 0\rangle \; ,$$
$$|\mathbf{7} \; ; \; -1, 1\rangle = F_1 |\mathbf{7} \; ; \; 1, -2\rangle \; ,$$
$$|\mathbf{7} \; ; \; 0, -1\rangle = F_2 |\mathbf{7} \; ; \; -1, 1\rangle \; ,$$
$$|\mathbf{1} \; ; \; 0, 0\rangle \; . \tag{A.2}$$

(There are two vectors $|\mathbf{14} \; ; \; 0, 0\rangle$, distinguished by a further degeneracy index.)

The invariant inner product on the representation space is defined by the conditions:

$$E_{1,2} = F^\dagger_{1,2} \; , H_{1,2} = H^\dagger_{1,2} \; , \tag{A.3}$$

and can be calculated as usual on complex vector spaces, but the conjugate of q must be understood by treating q as a formal variable, i.e. $q^* = q$. With respect to this product, we can compute $\mathcal{P}_\mathbf{R}$ for $\mathbf{R} = \mathbf{1}, \; \mathbf{7}, \; \mathbf{14}$ as the orthogonal projector on the corresponding subspace, while $\mathcal{P}_{\mathbf{27}}$ can be computed as the complementary projector (the irreducible subspaces are mutually orthogonal).

All these projectors commute with the quantum group action and therefore they provide us with the four linearly independent solutions of the intertwining equations for $\mathcal{U}_q(G_2)$.

B   The $\mathcal{U}_q(A_2)$ decomposition of the S-matrix

The transformation to the A_2 gradation is described by the formula

$$P_{12}\widehat{\mathcal{R}}(x, q)_{A_2} = x_1^{h_1 + 2h_2} \otimes x_2^{h_1 + 2h_2} P_{12}\widehat{\mathcal{R}}(x, q) x_1^{-h_1 - 2h_2} \otimes x_2^{-h_1 - 2h_2} \; . \tag{B.1}$$

In the A_2 gradation, the evaluation representation takes the form

$$\pi^{(\theta)}_{A_2}(e_0) = e_0 \; , \; \pi^{(\theta)}_{A_2}(e_1) = e_1 \; , \; \pi^{(\theta)}_{A_2}(e_2) = x^{1/3} e_2 \; ,$$
$$\pi^{(\theta)}_{A_2}(f_0) = f_0 \; , \; \pi^{(\theta)}_{A_2}(f_1) = f_1 \; , \; \pi^{(\theta)}_{A_2}(f_2) = x^{-1/3} e_2 \; ,$$
$$\tag{B.2}$$

so now we have a genuine $\mathcal{U}_q(A_2)$ algebra acting on the representation space $\mathbf{7}$, which decomposes into $\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ under this action.

Using this result, one can decompose $\widehat{\mathcal{R}}(x, q)_{A_2}$ into irreducible components. The notation $\widehat{\mathcal{R}}(x, q)^{\mathbf{cd}}_{\mathbf{ab}}$ means the component which maps from $\mathbf{c} \otimes \mathbf{d}$ to $\mathbf{a} \otimes \mathbf{b}$. The result is the following:

$$\widehat{\mathcal{R}}(x, q)^{\mathbf{11}}_{\mathbf{11}} =$$
$$\frac{(x - q^2)\left(x^2 q^{2/3} + x - 2xq^{2/3} + xq^2 - 2q^{8/3}x + q^{10/3}x - 2q^{14/3}x + q^{16/3}x + q^{14/3}\right)}{(x - q^{2/3})(x - q^4)(x - q^{8/3})}$$

$$\widehat{\mathcal{R}}(x, q)^{\mathbf{13}}_{\mathbf{13}} = \widehat{\mathcal{R}}(x, q)^{\bar{\mathbf{3}}\mathbf{1}}_{\mathbf{31}} = \frac{x^{2/3}\left(1 - q^{4/3}\right)\left(x - q^{2/3} + q^{4/3} - q^2\right)}{(x - q^{2/3})(x - q^{8/3})}$$

$$\widehat{\mathcal{R}}(x, q)^{\mathbf{1}\bar{\mathbf{3}}}_{\mathbf{13}} = \widehat{\mathcal{R}}(x, q)^{\mathbf{31}}_{\mathbf{31}} = \frac{x^{1/3}\left(1 - q^{4/3}\right)\left(x - xq^{2/3} + xq^{4/3} - q^2\right)}{(x - q^{2/3})(x - q^{8/3})}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{13}}_{\mathbf{31}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{31}}_{\mathbf{13}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{1\bar{3}}}_{\mathbf{\bar{3}1}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{\bar{3}1}}_{\mathbf{1\bar{3}}} = \frac{\left(x-q^2\right)\left(x-1\right)q^{2/3}}{\left(x-q^{2/3}\right)\left(x-q^{8/3}\right)}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{11}}_{\mathbf{33}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{\bar{3}\bar{3}}}_{\mathbf{11}} \frac{x^{1/3}\left(x-1\right)\left(q^{4/3}-1\right)\left(x-xq^{2/3}+xq^{4/3}-q^{10/3}\right)q^{4/3}}{\left(x-q^{2/3}\right)\left(x-q^4\right)\left(x-q^{8/3}\right)}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{11}}_{\mathbf{33}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{3\bar{3}}}_{\mathbf{11}} = \frac{x^{2/3}\left(x-1\right)\left(q^{4/3-1}\right)\left(x-q^2+q^{8/3}-q^{10/3}\right)q^{4/3}}{\left(x-q^{2/3}\right)\left(x-q^4\right)\left(x-q^{8/3}\right)}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{33}}_{\mathbf{33}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{\bar{3}\bar{3}}}_{\mathbf{33}} = \mathcal{P}_{\mathbf{6}} - \frac{q^{8/3}x - xq^2 + xq^{2/3} - x - xq^{4/3} + q^{4/3} + x^2q^2 - q^{10/3}x}{\left(x-q^{2/3}\right)\left(x-q^{8/3}\right)}\mathcal{P}_{\mathbf{3}}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{3\bar{3}}}_{\mathbf{33}} = \frac{x^{2/3}\left(1-q^{2/3}\right)}{x-q^{2/3}}\mathcal{P}_{\mathbf{8}} + \frac{x^{2/3}\left(q^{4/3}-1\right)f_1(x,q)}{\left(x-q^4\right)\left(x-q^{2/3}\right)\left(x-q^{8/3}\right)}\mathcal{P}_{\mathbf{1}}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{\bar{3}3}}_{\mathbf{33}} = \frac{x^{1/3}\left(1-q^{2/3}\right)}{x-q^{2/3}}\mathcal{P}_{\mathbf{8}} + \frac{x^{1/3}\left(1-q^{4/3}\right)f_2(x,q)}{\left(x-q^4\right)\left(x-q^{2/3}\right)\left(x-q^{8/3}\right)}\mathcal{P}_{\mathbf{1}}$$

$$\widehat{\mathcal{R}}(x,q)^{\mathbf{\bar{3}3}}_{\mathbf{33}} = \widehat{\mathcal{R}}(x,q)^{\mathbf{3\bar{3}}}_{\mathbf{33}} = \frac{q^{1/3}(x-1)}{x-q^{2/3}}\mathcal{P}_{\mathbf{8}} + \frac{(x-1)f_3(x,q)}{\left(x-q^4\right)\left(x-q^{2/3}\right)\left(x-q^{8/3}\right)}\mathcal{P}_{\mathbf{1}} \tag{B.3}$$

where we used the notation

$$f_1(x,q) = -q^6 + q^{16/3} - q^{14/3}x + xq^4 - q^{10/3}x + q^{10/3}x^2 + q^{8/3} -$$
$$q^{8/3}x^2 - q^2 + x^2q^2 + xq^2 - 2xq^{4/3} + q^{4/3} + xq^{2/3} - x$$
$$f_2(x,q) = xq^6 - q^{16/3}x - q^{14/3}x^2 + 2q^{14/3}x + q^4x^2 - xq^4 - q^4 +$$
$$q^{10/3} - q^{10/3}x^2 - q^{8/3} + q^{8/3}x - xq^2 + xq^{4/3} - x^2q^{2/3} + x^2$$
$$f_3(x,q) = (q^{8/3}x^2 + q^{14/3}x - xq^6 + q^{8/3}x + xq^{4/3} - 2xq^4 - 2xq^2 -$$
$$x + q^{10/3}x + q^{10/3})q^{2/3} \ . \tag{B.4}$$

$\mathcal{P}_{\mathbf{1}}$ and $\mathcal{P}_{\mathbf{8}}$ denote the projectors on the irreducible subspaces $\mathbf{1}$ and $\mathbf{8}$, while $\mathcal{P}_{\mathbf{3}}$ projects on $\mathbf{3}$ or $\mathbf{\bar{3}}$ and $\mathcal{P}_{\mathbf{6}}$ on $\mathbf{6}$ or $\mathbf{\bar{6}}$ depending on which occurs in the decomposition of the subspace indicated in the indices of $\widehat{\mathcal{R}}(x,q)$.

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